• Recall that for a surface $S$ parametrized by $G : D \rightarrow \mathbb{R}^3$ and a scalar valued function $f$ defined on $S$ then we can compute the scalar surface integral

$$\int\int_S f \, dS = \int\int_D f(G(u,v)) ||\mathbf{N}|| \, dA$$

where $\mathbf{N}$ is the cross products of the partial derivatives of $G$.

• The vector surface integral of a function $\mathbf{F}$ is defined to be the scalar surface integral of the normal component of $\mathbf{F}$ over $S$ as an oriented surface. It can be computed as

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int_D \mathbf{F}(G(u,v)) \cdot \mathbf{N} \, dS$$

where the normal vector is the cross product of the partial derivatives of $G$, pointing in the correct direction.

• Two notations for the line integral of a vector field:

$$\int_C \mathbf{F} \cdot d\mathbf{s} \quad \text{and} \quad \int_C F_1 \, dx + F_2 \, dy$$

where $\mathbf{F} = \langle F_1, F_2 \rangle$.

• The curl of a 2-dimensional vector field $\mathbf{F} = \langle F_1, F_2 \rangle$ is defined to be

$$\text{curl}_z(\mathbf{F}) = \boxed{0}.$$ 

• Green’s Theorem:

$$\oint_{\partial D} F_1 \, dx + F_2 \, dy = \boxed{0}$$

where $\partial D$ denotes the boundary of $D$ with its boundary orientation (counterclockwise).

• Formula for the area of the region $D$ enclosed by $C$:

$$\text{Area}(D) = \boxed{0}.$$
1. Use Green’s Theorem to evaluate $\int_C \left( \arctan(x^2) - y^2 \right) dx + \left( x^2 y - \log(y^2 + 1) \right) dy$, where $C$ is the semicircle $y = \sqrt{4 - x^2}$ together with the line segment from $(-2, 0)$ to $(2, 0)$ (oriented counterclockwise).

2. Recall that the **Laplace operator** is given by $\Delta \varphi = \varphi_{xx} + \varphi_{yy}$. For a vector field $F = \langle F_1, F_2 \rangle$, define the **conjugate vector field** by $F^* = \langle -F_2, F_1 \rangle$.

   (a) Show that if $F = \nabla \varphi$ then $\text{curl}_z(F^*) = \Delta \varphi$.

   (b) Let $n$ be the outward-pointing unit normal vector to a simple closed curve $C$. The **normal derivative** of a function $\varphi$, denoted $\partial \varphi / \partial n$ is defined to be $\nabla \varphi \cdot n$ (the directional derivative of $\varphi$ in the direction $n$). Show that

   $$\oint_C \frac{\partial \varphi}{\partial n} \, ds = \iint_D \Delta \varphi \, dA,$$

   where $D$ is the domain enclosed by $C$.

   (*Hints:* Recall that the outward pointing unit normal vector to a parametrized path $c(t) = (x(t), y(t))$ is given by $n(t) = (y'(t), -x'(t)) / \| c'(t) \|$. Let $F = \nabla \varphi$, show that $\partial \varphi / \partial n = F^* \cdot T$ where $T$ is the unit tangent vector to $C$, and then apply Green’s Theorem.)
Recall that for a surface $S$ parametrized by $G: D \rightarrow \mathbb{R}^3$ and a scalar valued function $f$ defined on $S$ then we can compute the scalar surface integral
\[
\iint_S f \, dS = \iint_D f(G(u,v)) \|N\| \, dA
\]
where $N$ is the cross products of the partial derivatives of $G$.

The vector surface integral of a function $F$ is defined to be the scalar surface integral of the normal component of $F$ over $S$ as an oriented surface. It can be computed as
\[
\iint_S F \cdot dS = \iint_D F(G(u,v)) \cdot N \, dS
\]
where the normal vector is the cross product of the partial derivatives of $G$, pointing in the correct direction.

Two notations for the line integral of a vector field:
\[
\int_C \mathbf{F} \cdot ds \quad \text{and} \quad \int_C F_1 \, dx + F_2 \, dy
\]
where $\mathbf{F} = \langle F_1, F_2 \rangle$.

The curl of a 2-dimensional vector field $\mathbf{F} = \langle F_1, F_2 \rangle$ is defined to be
\[
\text{curl}_z (\mathbf{F}) = \begin{vmatrix}
\frac{\partial}{\partial x} & - \frac{\partial}{\partial y} \\
\end{vmatrix}.
\]

Green’s Theorem:
\[
\oint_{\partial \mathcal{D}} F_1 \, dx + F_2 \, dy = \iint_{\mathcal{D}} \text{curl}_z (\mathbf{F}) \, dA
\]
where $\partial \mathcal{D}$ denotes the boundary of $\mathcal{D}$ with its boundary orientation (counterclockwise).

Formula for the area of the region $\mathcal{D}$ enclosed by $\mathcal{C}$:
\[
\text{Area}(\mathcal{D}) = \frac{1}{2} \oint_{\mathcal{C}} x \, dy - y \, dx.
\]
1. Use Green’s Theorem to evaluate \[ \int_C (\arctan(x^2) - y^2) \, dx + (x^2y - \log(y^2 + 1)) \, dy \], where \( C \) is the semicircle \( y = \sqrt{4 - x^2} \) together with the line segment from \((-2, 0)\) to \((2, 0)\) (oriented counterclockwise).

**Answer:** \( \frac{32}{3} \).

2. Recall that the Laplace operator is given by \( \Delta \varphi = \varphi_{xx} + \varphi_{yy} \). For a vector field \( F = \langle F_1, F_2 \rangle \), define the conjugate vector field by \( F^* = \langle -F_2, F_1 \rangle \).

(a) Show that if \( F = \nabla \varphi \) then \( \text{curl}_z (F^*) = \Delta \varphi \).

**Solution:** We have \( F = \langle \varphi_x, \varphi_y \rangle \), so
\[
\text{curl}_z (F^*) = \text{curl}_z \langle -\varphi_y, \varphi_x \rangle = (\varphi_x)_y - (\varphi_y)_x = \varphi_{xx} + \varphi_{yy} = \Delta \varphi.
\]

(b) Let \( \mathbf{n} \) be the outward-pointing unit normal vector to a simple closed curve \( C \). The **normal derivative** of a function \( \varphi \), denoted \( \partial \varphi / \partial \mathbf{n} \) is defined to be \( \nabla \varphi \cdot \mathbf{n} \) (the directional derivative of \( \varphi \) in the direction \( \mathbf{n} \)). Show that
\[
\int_C \frac{\partial \varphi}{\partial \mathbf{n}} \, ds = \oint_D \Delta \varphi \, dA,
\]
where \( D \) is the domain enclosed by \( C \).

**Hints:** Recall that the outward pointing unit normal vector to a parametrized path \( \mathbf{c}(t) = (x(t), y(t)) \) is given by \( \mathbf{n}(t) = \langle y'(t), -x'(t) \rangle / \| \mathbf{c}'(t) \| \). Let \( F = \nabla \varphi \), show that \( \partial \varphi / \partial \mathbf{n} = F^* \cdot T \) where \( T \) is the unit tangent vector to \( C \), and then apply Green’s Theorem.

**Solution:** Let \( \mathbf{c}(t) \) be a parametrization of \( C \). If \( F = \nabla \varphi = \langle \varphi_x, \varphi_y \rangle \), then \( F^* = \langle -\varphi_y, \varphi_x \rangle \), and we have
\[
\frac{\partial \varphi}{\partial \mathbf{n}} = \nabla \varphi \cdot \mathbf{n}
= \langle \varphi_x, \varphi_y \rangle \cdot \left( \frac{\langle y'(t), -x'(t) \rangle}{\| \mathbf{c}'(t) \|} \right)
= \frac{1}{\| \mathbf{c}'(t) \|} (\varphi_x y'(t) - \varphi_y x'(t))
= \frac{1}{\| \mathbf{c}'(t) \|} (-\varphi_y x'(t) + \varphi_x y'(t))
= \langle -\varphi_y, \varphi_x \rangle \cdot \left( \frac{\langle x'(t), y'(t) \rangle}{\| \mathbf{c}'(t) \|} \right)
= F^* \cdot T.
\]
(Check that you can justify each equality above.) Now, applying Green's Theorem to the vector field \( \mathbf{F}^* \) and using part (a), we obtain

\[
\oint_C \frac{\partial \varphi}{\partial \mathbf{n}} \, ds = \oint_C \mathbf{F}^* \cdot \mathbf{T} \, ds = \iint_D \text{curl}_z \mathbf{F}^* \, dA = \iint_D \Delta \varphi \, dA,
\]

which is what we wanted to show.